Randomized Algo: Closest Pair

def: Given $P_1 = (x_1, y_1) \dots P_n(x_n, y_n) \in \mathbb{Z}^n$: The closest pair is to find

$$\arg\min_{i \neq j} \frac{dist((x_i, y_i), (x_j, y_j))}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}}$$

Conventions: All numbers $O(\lg n)$ bits.

$$|x_i|, |y_i| \leq N = poly(n)$$

Thm (prev) Closest pair in $O(n \lg n)$ expected time.

Thm in $O(n)$ expected time.

idea: data structure $\Rightarrow$ efficient way to store data.

Warmup Can we verify minimum dist $\geq \Delta$, $\Delta \in \mathbb{Z}$?

idea: process points in order.

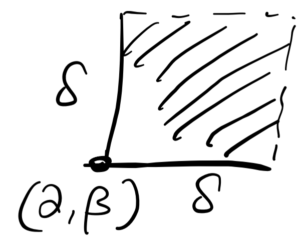
def: $\Delta_k = \min_{i,j \leq k} \text{dist}(p_i, p_j)$

Q: Compare Δ_n v.s. Δ ? Δ_{k-1} v.s. Δ_k ?

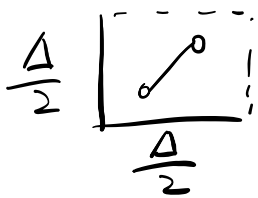
idea: coarse discretization of space

def: $\delta > 0$, $\delta \in \mathbb{R}$, a δ-subsquare is a subset of $\mathbb{Z}^2$ given

$$\boxed{S_{\underbrace{\alpha,\beta}_{\mathbb{R}}} = \left\{ (x,y) \in \mathbb{Z}^2 : \begin{array}{l} \alpha \leq x \leq \alpha + \delta \\ \beta \leq y \leq \beta + \delta \end{array} \right.}$$



<u>claim</u>: $\Delta_k \geq \Delta \implies$ each $\Delta/2$ subsquare have ≤ 1 points from p_i to p_k.

Pf: 

$\implies$ distance $\leq \Delta/\sqrt{2} < \Delta$ (contrapositive)

Idea: round coordinates:

def: $(a,b) \in \mathbb{N}^2$, the $(a,b)\frac{\Delta}{2}$ subsquare of $\mathbb{N}^2$

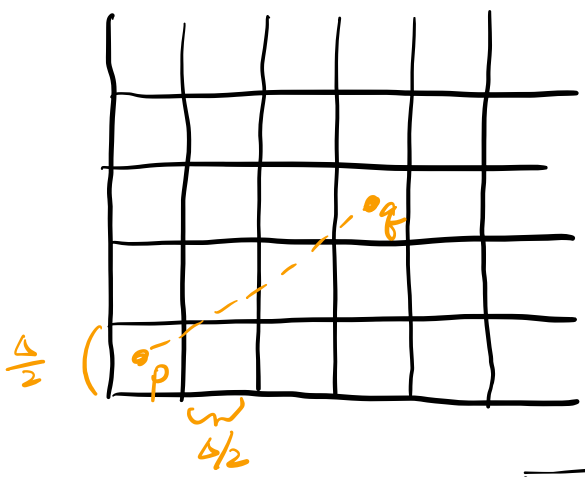
is $S_{a\cdot\Delta/2,\, b\cdot\Delta/2}$ Partition

The $\frac{\Delta}{2}$-grid is $\{S_{a,b},\ 0 \le a,b \le \frac{2N}{\Delta}\} = T_{a,b}$

prop $d(p,q) \le \Delta$, $p \in S_{a,b}$, $q \in S_{c,d}$

$\Rightarrow |a-c|, |b-d| \le 2$

Pf: By contrapositive:



If $|a-c| > 2 \Rightarrow dist(p,q) = \sqrt{(x-z)^2 + (y-w)^2} \ge |x-z| > \Delta$

def A dictionary over $\mathcal{U}$ is a data structure for a set $S \subseteq \mathcal{U}$ of keys x along with value y.

Supports $\begin{cases} \text{insert}(x, y) \\ \text{lookup}(z) \end{cases}$

Algo:

- Init dictionary A on $\mathcal{U} = \left\{ (a,b),\ 0 \le a, b \le \frac{2N}{\Delta} \right\}$
- for $1 \le i \le n$
  - $(a,b) = \left(\left\lfloor \frac{x_i}{\Delta/2} \right\rfloor, \left\lfloor \frac{y_i}{\Delta/2} \right\rfloor \right)$ round down
  - Compute $\Delta_i' = \min d(p_i, p)$, $p \in A[c,d]$, $|a-c|, |b-d| \le 2$.
  - if $\Delta_i' < \Delta$, return p_i, p, Δ_i'
  - Insert p_i into $A[a,b]$
- return $\Delta_n \ge \Delta$

Prop Suppose $\Delta_i \geq \Delta$, then

(a) algo reach insertion

(b) $A[a,b]$ empty when reach insertion

Cor Suppose $\Delta_i \geq \Delta$ then ≤ 25 lookups and return ≤ 25 points.

Thm $S \subseteq U$ $\quad |S| = n$, $\quad |U| \leq N = \mathrm{poly}(n)$

one can in deterministic $O(n)$ time, construct hash family $\mathcal{H}: U \to T$

- $|T| \leq O(n)$
- Choose $h \in \mathcal{H}$ takes $\begin{cases} O(1) \text{ space.} \\ O(1) \text{ eval} \end{cases}$
- insertions, lookup in $O(1)$ expected.

Cor: If $\Delta_n \geq \Delta$ then $\begin{cases} \text{- algo is correct} \\ \text{- run in } O(n) \text{ expected.} \end{cases}$

Cor $\exists k$, $\Delta_{k-1} \geq \Delta$, $\Delta_k < \Delta$

Then algo is correct, returns $\Delta' = \Delta_k$

Runs in $O(k)$ expected time.

Use this with random order $P_1 \dots P_n$

$(p,q) = (P_1, P_2)$

$\Delta = d(p,q)$

while

  verify $\Delta_n \geq \Delta$

  if $\Delta_n \geq \Delta$

    return (p,q)

  else

    $(p,q), \Delta \leftarrow (p',q'), \Delta'$

$\Rightarrow$ correct

Prop Algo update some $\Delta \to \Delta'$ once

def $I_k = \begin{cases} 1 & \text{verification algo updates } \Delta \to \Delta_k \\ 0 & \text{o.w.} \end{cases}$

prop # dict operations is $O(n + \sum_{k=1}^{n} k \cdot I_k)$

pf: $O(n)$ ops in final pass

$O(k)$ ops to update $\Delta \to \Delta' = \Delta_k$ happen iff $I_k = 1$

Key: What is $\Pr[I_k = 1]$?

prop $\Pr[I_k = 1] \leq \frac{2}{k}$

pf: $I_k = 1 \iff$ update $\Delta \to \Delta_k \iff \Delta_1, \dots, \Delta_{k-1} > \Delta$

$\iff \min\limits_{i,j<k} d(p_i, p_j) > \min\limits_{i<k} d(p_i, p_k)$

$\Rightarrow$ in a random ordering, in the first k points, $\exists$ 2 points that are close (will update $\Delta \to \Delta_k$). The probability that one of these two stays in the end is $\frac{2}{k}$.

$\Rightarrow \le \frac{2}{k}$

Cor algo in $O(n)$ expected

$$\mathbb{E}_0 \quad \mathbb{E}_A \Big[\underset{\substack{\uparrow \\ O(n)}}{D} + \sum_k \underset{\substack{\uparrow \\ O(k)}}{D_k} \cdot I_k \Big]$$

$$= \mathbb{E}_0 \Big\{ \Big[\underbrace{\mathbb{E}_A[D]}_{O(n)} + \sum_k \mathbb{E}_A [D_k \cdot I_k] \Big] \Big\}$$

not related to A

$\Downarrow$

$\mathbb{E}_A[D_k]$ $\leftarrow$ $O(k)$ ops, k random variable independent from A.

$= I_k \, O(k)$

$$= \mathbb{E}_0 \left\{ O(n) + \sum I_k O(k) \right\} = O(n) + \sum O(k) \Pr[I_k = 1]$$

$$\leq O(n) + \sum_{k=1} O(k) \cdot \frac{k}{2}$$

$$\leq O(n)$$