Randomized Algo: Min Cut

def $G = (V, E)$ undirected

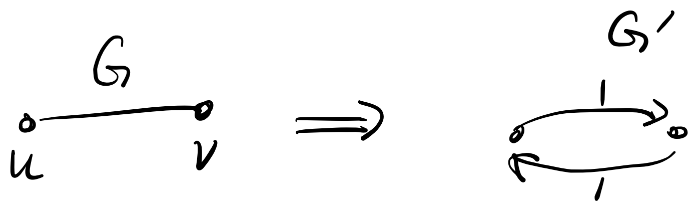
A cut $V = A \cup B$, $A, B \neq \emptyset$.

The capacity $|C| = \left| \{ e : (u,v), u \in A, v \in B \} \right|$

The global min cut problem v.s. (s,t) min cut problem. ?

prop: Let $G = (V, E)$ undirected. $G' = (V', E')$ be capacitated directed graph.

$E' = \{ e, -e, e \in E \}$ and capacity $c_e = 1$, $V' = V$



Any cut C w/ $V = \underset{\ni s}{A} \cup \underset{\ni t}{B}$, $|C| = |C|$

in G, global cut

in G' (s,t)-cut

Pf:

prop: Global mincut in $O(n^2 m)$ time.

Pf: algo:

- construct G' from G.
- $s = v_1$ → just some vertex v_1
- for $t \leftarrow v_2, \dots, v_n$
  compute min (s,t)-cut in G'
- return the minimum of for-loop above.

Complexity: n times of Orlin's Algo. ⇒ $O(n) \cdot O(nm)$

Correctness:

(1) claim: outputs is a valid cut. ⇒ by Orlin

(2) claim: $C = A \cup B$ is the global mincut in G.

Pf(2) either

- $v_1 \in A,\ v_j \in B,\ j \neq 1 \Rightarrow C$ is (v_1, v_j)-cut.
  $\Rightarrow C$ is found.

- $v_1 \notin A \Rightarrow v_1 \in B, v_j \in B, j \neq 1 \Rightarrow C = B \cup A$ is (v_1, v_j)-cut. Just reverse

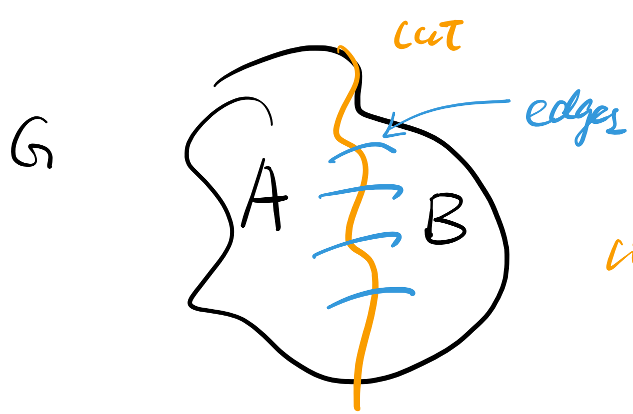
Q: Global v.s. (s,t) min-cut comparison? Which one is worse?

Thm: Global min cut in $O(n^4 \lg n)$ w/ high probability,

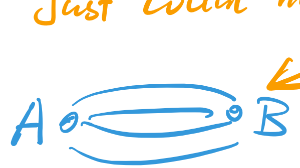
Remark:
- $O(n^4 \lg n)$ worse than $O(n^2 m)$
- but algo is simple $\Rightarrow$ can reduce to $O(n^2 \lg n)$

Thm: Global min cut in $O(n^4 \lg n)$

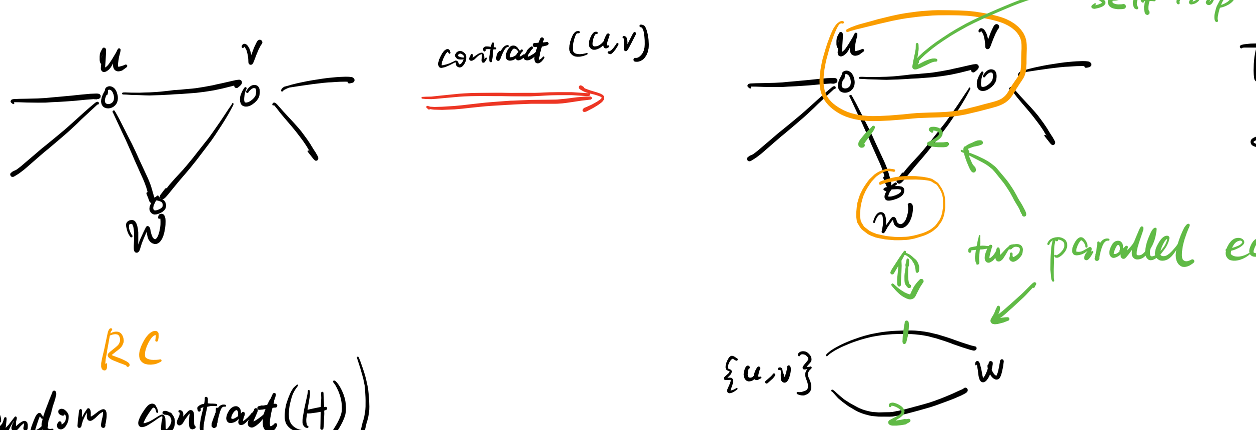
idea: contract a random edge.



contract



Ex



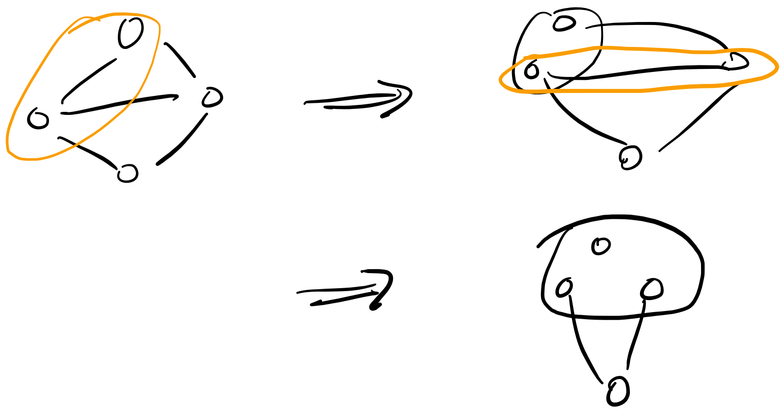
Treat $\{u, v\}$ as a supper vertex

RC

Algo(random contract(H))

- global $G=(V,E)$
- H multigraph:
  - vetices $S_1, \cdots, S_k$, $S_i \subseteq V$, $\underbrace{V = S_1 \cup S_2 \cup \cdots \cup S_k, \; S_i \cap S_j = \emptyset \text{ for } i \neq j}_{\text{patition of } V}$
  - edges between S_i, S_j: $e = (u,v)$, $u \in S_i$, $v \in S_j$
- If $k=2$, return $S_1 \cup S_2$
- Else, randomly contract an edge e:
  - choose e uniformly between S_i, S_j
  - merge $S_i, S_j \Rightarrow S_i \cup S_j$
  - delete self loops
  - return random contraction(H') → recurse.

Ex)



4 edges

two super vertices.

Fact: Contraction Algo can be implemented in $O(n^2)$

Q: Success probability?

prop: RC returns a global mincut C is $\frac{1}{\binom{n}{2}}$

Rmk: There are $2^n - 2$ possible cuts

trival cuts $\{(S=\phi, T=V), (S=V, T=\phi)\}$

Q How can RC fail?

prop: $C = A \cup B$ output iff no edge between A and B never get contracted.

Pf: $(\neg \Leftarrow \neg)$ if $e=(u,v)$, $u \in A$, $v \in B$ is contracted.

then e is deleted.

Also, the final cut is all edges between $(S_1, S_2) \Rightarrow (S_1, S_2) \neq (A, B)$

($\Leftarrow$) By induction: $A = T_1 \cup \cdots \cup T_l$, $B = T_1' \cup \cdots \cup T_{l'}'$

Next contraction $e:(u,v) \Rightarrow \begin{cases} e \text{ between } T_i, T_j \\ e \text{ between } T_i', T_j' \end{cases}$

$\Rightarrow$ $A \cup B$ is a valid cut in contracted multigraph.

$\Rightarrow$ In the end, $(S_1, S_2) = (A, B)$ $\square$

Q $\Pr\{\text{no edge of cut } C \text{ is contracted}\}$?

prop What is $\Pr\{\text{no edge of } C \text{ is contracted in the first round?}\}$

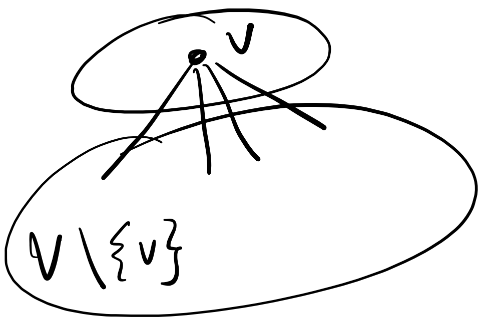
$$= 1 - \frac{|C|}{|E|}$$

Q What is $\frac{|C|}{|E|}$?

prop Multigraph $H = (V, E)$ global mincut size $\overset{|C|}{\overset{\|}{r}}$ $\Rightarrow |E| \geq \frac{rn}{2}$

Pf: Global mincut $\Rightarrow$ no matter how to choose cut, the cut is at least r.

Consider



for all v, at least $\deg(v) \geq r$

$\Rightarrow 2|E| = \sum_v \deg(v) \geq n \cdot r$

$|E| \geq \frac{nr}{2}$ $\blacksquare$

Cor $\Pr[\text{No edge is contracted in first round}] \geq 1 - \frac{|C|}{\frac{|C| \cdot n}{2}} = 1 - \frac{2}{n}$

In the first round unlikely to fail.

prop $\mathcal{E}_j$ = event that no edge in C is contracted in round j

$\Pr\{\mathcal{E}_1\} \geq 1 - \frac{2}{n}$. $\Pr\{\mathcal{E}_{j+1} \mid \mathcal{E}_1 \wedge \mathcal{E}_2 \wedge \cdots \wedge \mathcal{E}_j\} \geq 1 - \frac{2}{n-j}$

Pf No edge of C contracted in first j round

$\Rightarrow$ multigraph H

- respects partition A, B, $\begin{cases} A = T_1 \cup \cdots \cup T_\ell \\ B = T_1' \cup \cdots \cup T_{\ell'}' \end{cases}$
- have $n-j$ vertices (each time $- = 1$

$\Rightarrow$ A, B is a valid cut with value $|C|$

Claim: any cut in H represents a cut in G

Pf sketch: undo merges.

Cor H has mincut value $|C|$

$\Rightarrow$ H has $\geq \underset{r}{|C|} \cdot \frac{\overset{\#\text{vert}}{(n-j)}}{2}$ edges $\Rightarrow \Pr\{\sim\} \geq 1 - \frac{|C|}{|C| \cdot \frac{(n-j)}{2}} = 1 - \frac{2}{n-j}$ ■

Cor $\Pr\{\text{no edge in } C \text{ is ever contracted}\} \geq \frac{1}{\binom{n}{2}}$

Can this be shown using combinatorics?

$$\Pr\{\text{no edge in } C \text{ is ever contracted}\} = \Pr\{\mathcal{E}_1 \cap \mathcal{E}_2 \cap \cdots \cap \mathcal{E}_{n-2}\}$$

Pf:

$$= \Pr\{\mathcal{E}_1\} \cdot \Pr\{\mathcal{E}_2 \mid \mathcal{E}_1\} \cdot \Pr\{\mathcal{E}_3 \mid \mathcal{E}_1 \cap \mathcal{E}_2\} \cdots \Pr\{\mathcal{E}_{n-2} \mid \mathcal{E}_1 \cap \mathcal{E}_2 \cdots \cap \mathcal{E}_{n-3}\}$$

$$\geq \left(1 - \frac{2}{n}\right) \cdot \left(1 - \frac{2}{n-1}\right) \cdot \cdots \cdot \left(1 - \frac{2}{n-(n-3)}\right)$$

$$= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \cdots \cdot \frac{n-(n-1)}{n-(n-3)} = \frac{1}{3}$$

$$= \frac{(n-2)!}{n! \cdot 2} = \frac{1}{\binom{n}{2}} \qquad \blacksquare$$

Q: Isn't this terrible? idea: amplify probability of success by repetition.

Algo:
- Randomly contract G k times
- Return best cut.

Cor Let's fix some mincut C in G. In k random contraction, we will observe C except w/ probability $\leq \left(1 - \frac{1}{\binom{n}{2}}\right)^k$

Cor $O(n^2 \lg n)$ random contraction will produce mincut w/ high probability.

Pf: $k = \binom{n}{2} \lg n \Rightarrow \left(1 - \frac{1}{\binom{n}{2}}\right)^k = \left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2}\lg n} \le \left(e^{-\frac{1}{\binom{n}{2}}}\right)^{\binom{n}{2}\lg n} = e^{-\lg n} = \frac{1}{n}$

again $1+z \le e^z$

Thm Global mincut in $O(n^4 \lg n)$ steps w/ high probability.

Sketch: $O(n^2 \lg n)$ $\underbrace{\text{random contraction}}_{O(n^2)}$.