Linear Programming

Reduction:

mincut $\le$ maxflow
bipartite $\le$ maxflow
shortest path

$\le \boxed{?}$

is there a universal efficient algo that we can reduce to?

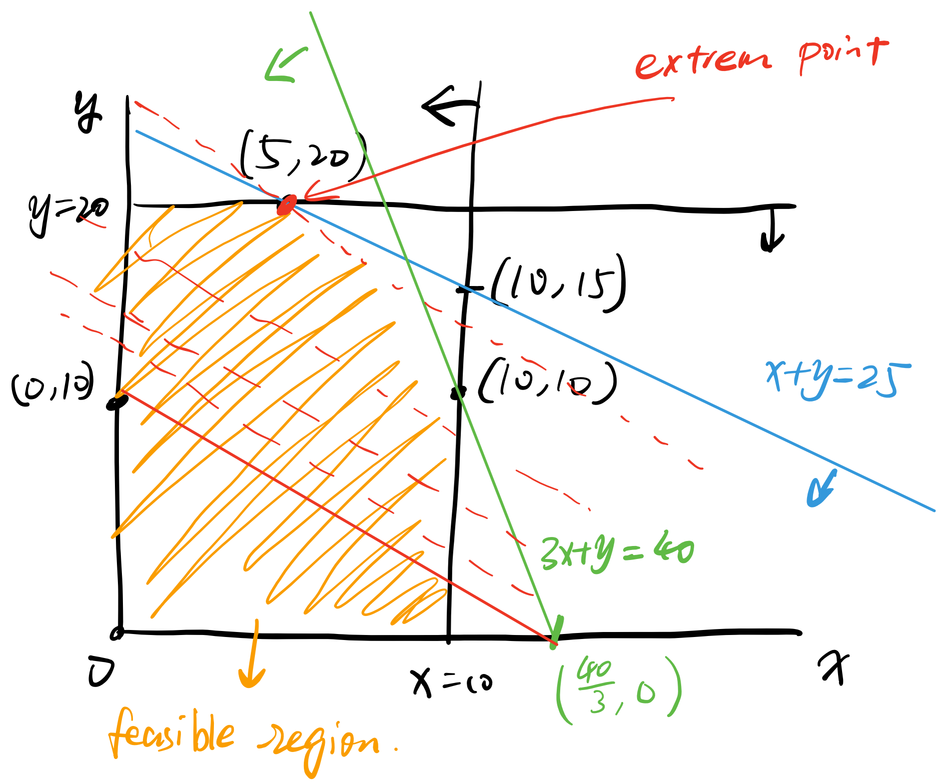
Ex New tea shop
- black tea
- green tea

$\Rightarrow$ how much perday?

$x \leftarrow$ # black created / day
$y \leftarrow$ # green —— / day

Constraints
- $x \le 10$
- $y \le 20$
- $x+y \le 25 \le 30$ $\leftarrow$ bubble constraint
- $3x+y \le 40$ $\leftarrow$ suggar constraint

Goal: maximize $6x + 8y$



Rmk : - The shaded region is continuous region $\Rightarrow$ no obvious finite time algo.

- Can be high dimension

- The region is polyhydron

def A linear program (LP) is

$c \in \mathbb{R}^n$ (coefficient that defines objective function)

$A_\leq, A_=, A_\geq \in \mathbb{R}^{m \times n}$

$b_\leq, b_=, b_\geq \in \mathbb{R}^m$

$$|\Pi| = \max_x \langle c, x \rangle \text{ s.t. } \begin{cases} A_\leq \cdot x \leq b_\leq \\ A_= \cdot x = b_= \\ A_\geq \cdot x \geq b_\geq \end{cases}$$

objective function

The input size is :

$n = \#$ var

$m = \#$ constraints.

m is not bounded by n.

def

Say Π is feasible if $\exists\ x \in \mathbb{R}^n$ that satisfies constraints.

else infeasible

Π is bounded if $|\Pi| < \infty$. Otherwise unbounded.

Q: Given Π, what is $|\Pi|$?

LP
- linear objective functions
- linear constraints

Philosophy: maximize ~ minimize $\Longleftrightarrow$ $\min \langle c,x\rangle = -\max\langle -c,x\rangle$

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def A canonical form of LP:

$$\max \ \langle c,x\rangle \quad \text{s.t. } Ax \le b,\ x \ge 0$$

restrict to this def

lem No loss of generality using this restriction.

$\exists$ efficient maps $\begin{cases} x \mapsto x' \\ x' \mapsto x \end{cases} \Rightarrow$ $\underbrace{x \text{ is fesible in } \Pi \Leftrightarrow x' \text{ is feasible in } \Pi'}$

$\Downarrow$

$|\Pi| = |\Pi'|$

Pf: $A_{\geq} \cdot x \geq b_{\geq} \equiv (-A_{\geq}) \cdot x \leq (-b_{\geq})$

$$A_{=} \cdot x = b \Rightarrow \begin{bmatrix} A_{=} \\ \hline -A_{=} \end{bmatrix} x \leq \begin{bmatrix} b_{=} \\ \hline -b_{=} \end{bmatrix}$$

Remains: force $x \geq 0$

Create x' as $x^+, x^- \in \mathbb{R}^n_{\geq 0}$ by
$$\begin{cases} x_i^+ = \begin{cases} x_i & \text{if } x_i \geq 0 \\ 0 \end{cases} \\ x_i^- = \begin{cases} -x_i & \text{if } x_i \leq 0 \\ 0 \end{cases} \end{cases}$$

$$\Rightarrow x' = \underset{\geq 0}{x^+} - \underset{\geq 0}{x^-}$$

Define A' by $Ax = Ax^+ - Ax^- \leq b$

$$= \underbrace{[A \,\vdots\, -A]}_{A'} \underbrace{\begin{bmatrix} x^+ \\ \hline x^- \end{bmatrix}}_{x'} \leq \underbrace{\begin{bmatrix} | \\ b \\ | \end{bmatrix}}_{b'}$$

$\Rightarrow$ now $x' \geq 0$

Define c' by $\langle c, x \rangle = \langle c, x^+ \rangle - \langle c, x^- \rangle = \langle c', x' \rangle$

Another direction:

$A(x^+ - x^-) \leq b, \Rightarrow x = x^+ - x^- \Rightarrow Ax \leq b \Rightarrow \langle c, x \rangle = \langle c', x' \rangle$

Q: Maxflow vs. LP ?

prop: G capacitated graph, s, t, $c_e \geq 0$

$\boxed{\text{maxflow in } G}$ =

$$\max \quad f(s) = \boxed{\sum_{e: s \to \bullet} f_e - \sum_{e: \bullet \to s} f_e}$$ linear objective

$$\text{s.t.} \quad \forall e, \ f_e \geq 0, \ f_e \leq c_e.$$

$$\forall v \neq s, t, \quad f(v) = 0$$

$$\Downarrow$$

$$\boxed{\sum_{e: v \to \bullet} f_e - \sum_{e: \bullet \to v} f_e}$$ also linear

Express mincut in LP:

$$\boxed{\min_{\substack{V = S \cup T \\ s \in S,\ t \in T}} \underbrace{|C(S,T)|}_{\substack{\| \\ \sum\limits_{\substack{e: u \to v \\ u \in S,\ v \in T}} c_e}}} = \min \sum_e c_e \cdot x_e$$

$$\text{s.t.} \quad d_s = 0,\ d_t = 1$$

$$\forall e: u \to v \quad d_v \le d_u + x_e$$

Show they have same minimum:

$(\geq)$ given $V = S \cup T$, define $d_v = \begin{cases} 0 & \text{if } v \in S \\ 1 & \text{if } v \in T \end{cases} \Rightarrow d_s = 0,\ d_t = 1$

define $x_e = \begin{cases} 1 & \text{if } e: u \to v,\ u \in S,\ v \in T \\ 0 & \text{o.w.} \end{cases}$

claim: $\forall e: u \to v$,

d_v	$\le d_u +$	x_e
0	0	0
0	1	0
1	0	1
1	1	0

Pf: four cases:

$\Rightarrow$ $(d_v)_v$, $(x_e)_e$ feasible if

$$C(S,T) = \sum_{e: u \to v} c_e \cdot 1$$

$$= \sum_e c_e \cdot x_e = \langle c, x \rangle$$

$\Rightarrow |\Pi| \le$ mincut

($\le$) given d, x optimal (possibly not integral)
(This is the LP)

idea: randomized rounding

Algo: choose $\theta \in [0,1]$ randomly

define $S = \{ v : d_v < \theta \}$

output $V = S \cup T$

claim $s \in S$, $t \in T$.

claim: $\mathbb{E}[\,|C(S,T)|\,] \le |\Pi|$

cor: $\exists\ V = S \cup T$,

$C(S,T) \le \mathbb{E}_\theta[\,|C(S,T)|\,] \le |\Pi|$

pf: $\mathbb{E}[\,|C(S,T)|\,] = \mathbb{E}[\sum_{e : u \to v}$