LP

canonical form

def primal LP $\Pi = \max \langle c, x\rangle$ s.t. $Ax \le b$, $x \ge 0$ $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$

n variables.

m non-axess constraints.

has dual $\amalg$

$\min \langle b, y\rangle$ ← m variables

s.t. $A^T y \ge c$ ← n non-axess constraints

$y \ge 0$

rmk dual is not in canonical form.

Thm (weak duality) $|\Pi| \le |\amalg|$ if both fesible.

$|\Pi| = \max\langle c,x\rangle$, $|\amalg| = \min\langle b,y\rangle$

Pf: $x \ge 0$ fesible for $\Pi \Rightarrow Ax \le b$

$y \ge 0$ fesible for $\amalg \Rightarrow A^T y \ge c \Rightarrow y^T A \ge c^T \Rightarrow y^T A x \ge c^T x = \langle c, x\rangle$

$y^T A x \le y^T b = \langle b, x\rangle$

Cor dual is unbounded $\Leftrightarrow$ ($|\amalg| = -\infty \Rightarrow$ primal infeasible)

primal is unbounded $\Leftrightarrow$ ($|\Pi| = \infty \Rightarrow$ dual infeasible)

Thm (strong duality)

If Π feasible and bounded, then $\amalg$ feasible and bounded and $|\Pi| = |\amalg|$

Thm (weak duality)

$$|\Pi| = \max_{\substack{\text{s.t. } Ax = b \\ Ax' \le b' \\ x \ge 0}} \langle c, x\rangle \quad \le \quad \min_{\substack{A^T y + (A')^T z \ge c \\ z \ge 0 \\ y \text{ unrestricted}}} \langle b, y\rangle + \langle b', z\rangle$$

Pf:

$$\langle c, x\rangle = x^T c \le x^T\left(A^T y + (A')^T z\right)$$

$$= \underbrace{x^T A^T}_{b^T} y + \underbrace{x^T (A')^T}_{\le (b')^T} z$$

TODO

Now, back to maxflow, mincut

prop: maxflow $= \max \sum_{e:s\to\cdot} f_e - \sum_{e:\cdot\to s} f_e$

s.t. $f_{out} - f_{in} = 0$, $f_e \le c_e$, $f_e \geqslant 0$

has dual $\min \sum_e c_e x_e$

s.t. $\begin{cases} d_v \le d_u + x_e, & e: u\to v, \forall e. \\ d_s = 0, \ d_t = 1, & x_e \geqslant 0 \ \forall e. \end{cases}$

Pf: Rewrite maxflow LP

$$\max \quad -F_t$$

$$\left.\begin{array}{l} d_s \left(\sum_{e:s\to\cdot} f_e - \sum_{e:\cdot\to s} f_e - F_s = 0 \right) \\ d_t \left(\sum_{e:t\to\cdot} f_e - \sum_{e:\cdot\to t} f_e - F_t = 0 \right) \\ d_v \left(\sum_{e:v\to\cdot} f_e - \sum_{e:\cdot\to v} f_e = 0 \right) \\ x_e \left(\begin{cases} f_e \le c_e \\ f_e \geqslant 0 \end{cases} \right) \end{array}\right\} \text{sum up}$$

$$\Rightarrow -F_t \le \min \underbrace{d_s\cdot 0 + d_t \cdot 0 + \sum_v d_v \cdot 0 + \sum_e c_e x_e}_{= \sum_e c_e x_e}$$

s.t. $d_s = 0$, $d_t = 1$, $\underbrace{d_u - d_v + x_e \geqslant 0}_{= \ d_u \geqslant d_v + x_e}$, $e: u\to v$

Cor strong duality $\Longrightarrow$ max flow = mincut

rmk When use FF, we need integral capacity to prove

Now, we prove the equality works for any capacity.