Linear Program

High level of last lecture (L20)

- LP of m constant and n vars solved in $(m+n)^{O(2^n)}$

sketch: $\Pi = \max \langle c, x\rangle$ s.t. $Ax \le b$, $x \ge 0$ $\xrightarrow{\text{form}}$ polyhedron $P = \left\{ (\underset{\mathbb{R}^n}{x}, \underset{\mathbb{R}}{z}) : \begin{array}{l} z \le \langle c, x\rangle \\ Ax \le b \\ x \ge 0 \end{array} \right\}$

$\Rightarrow$ use Forier Motzkin elimination to compute equations for $P_z = \Pi_z(P)$

$\Rightarrow$ inspect $P_z \Leftrightarrow \alpha z \le \beta$
$$\begin{cases} 0z \le -1 & \Rightarrow z \in \emptyset \quad \text{infeasible} \\ 0z \le 0 & \Rightarrow z \in \mathbb{R} \quad \text{unbonded} \\ z \le \frac{\beta}{\alpha} & \Rightarrow z \in (-\infty, \frac{\beta}{\alpha}) \end{cases}$$

- Complexity: for each elimination, introduce $(m+n+1)^2$ blowup

$\Rightarrow$ $O(n)$ variable $\Rightarrow$ $(m+n)^{O(2^n)}$

Better : polynomial (m,n) ?

internal : $2^{poly(m,n)}$?

Q what do opt looks like ?

Q Is opt unique ? Finitely many ? Infinitely man ?

Q LP on integers ? Need infinite precision ?

Q Intersections on higher dimension ?

Cramer's Rule.

$Ax=b$, A invertible $\Rightarrow$ x solvable in $O(n^3)$

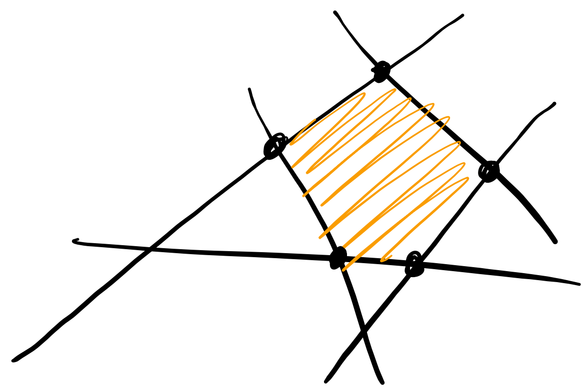
Defining a vertex of polyhedron:

def $P = \{x : Ax \le b\}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$

Given $x \in P$, an $(Ax)_i \le b_i$ is tight if $(Ax)_i = b_i$

A vertex of P is point $x^* \in P$ where $\exists\ S \subseteq [m]$

- $|S| = n$
- $(Ax^*)_i = b_i \quad \forall i \in S$
- $\det(A|_S) \neq 0 \Rightarrow x^*$ is unique.



cor $(m+n)^{O(n)}$ for solving LP

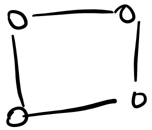
Given $\Pi = \max \langle c, x \rangle$ Algo: Try all vertices

s.t. $Ax \le b$

$x \ge 0$

$x \le B$

$$\binom{m+2n}{n}$$



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$\# all = 3^4$

e_1 has 3 choices.

e_2 and e_4 has 2 choices. $\begin{cases} \text{same choice} \Rightarrow 2 \text{ choices for } e_3 \\ \text{diff choice} \Rightarrow 1 \text{ for } e_3 \end{cases}$

Let $X_i = \begin{cases} 1 & \text{if } i \text{ is valid} \\ 0 & \text{o.w.} \end{cases}$

$$Y = \sum_{i=1}^{n} X_i$$

$3 \times (2\times2 + 2\times1) =$

$$3\times(4+2) = \frac{18}{\cancel{3\times3\times}\,3\times3} = \frac{2}{9}$$

$$\mathbb{P}[Y \geq \tfrac{4}{9}n]$$

$\mathbb{E}[X_i] = \frac{2}{9}$, $\mathbb{E}[Y] = n \cdot \mathbb{E}[X] = \frac{2}{9}n.$

$\mathbb{P}[Y \geq 2 \cdot \mathbb{E}[Y]] \leq \frac{1}{2}$ by markov.

$$\mathbb{P}[Y \geq (1+1)\,\mathbb{E}[Y]] \leq e^{-\mathbb{E}[Y]/4} = e^{-n/18}$$

$$\frac{-\frac{2}{9}n}{4} \qquad \frac{1}{18}$$

(c) For each round, there are $p = \frac{7}{9}$ probability to be removed.

$$\left(\frac{2}{9}\right)^k$$

(d) WTS: $\mathbb{E}[\#\text{round}] = O(\log n)$

$$= \sum x \cdot \mathbb{P}(\#r = x)$$

$\mathbb{P}(\#\text{rounds} \geq \log n) = \mathbb{P}(\exists i \text{ s.t. } i \text{ survives } \log n) \text{ rounds.}$

$\leq n \cdot \mathbb{P}(\text{one survives } \log n).$

$=$

$n \cdot (\quad)$