Graph & DP

def $G = (V, E)$ directed simple graph

$|V| = n \quad |E| = m$

cost function $C : E \to \mathbb{Z}$

def A path in G is a sequence of vertices $v_1, \dots, v_k$ s.t. $(v_i, v_{i+1}) \in E \ \forall i$

def is a cycle if $v_1 = v_k$

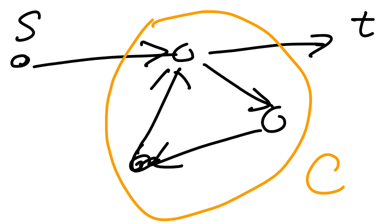
def The cost of path p is $|p| = \sum_{e \in p} C(e)$

def The distance

$dist(s,t) = \min_{p: s \leadsto t} |p|$, the argmin is shortest path

Lemma: If $\exists$ path $s \rightsquigarrow t$, sharing a vertex with a cycle C, where $|C| < 0$

then $dist(s,t) = -\infty$

pf: 

$\Longrightarrow$ the distance is not bounded (wait, path? walk?)

Lemma:

If no $s \rightsquigarrow t$ path touching neg cycle, then shortest path exists, with length $\le n-1$

pf:

claim: for any finite $p: s \rightsquigarrow t$, $\exists$ p', $s \Rightarrow t$ with $n-1$ edges, s.t. $|p'| \le |p|$

pf: (1) path q with $\ge n$ edges $\Rightarrow$ q has repeated vertex. By pigeonhole.

(2) $q: s \Rightarrow t$ w/ $k \ge n$ edges, then

$\exists$ $s \Rightarrow t$ q' w/
- $< k$ edges
- $|q'| \le |q|$

Note: sometimes, greedy algo won't work (Dijkstra). $\Longrightarrow$ DP!

def: $dist_{\le k}(s,v) = \min_{\substack{p: s \rightsquigarrow t \\ p \text{ has } \le k \text{ edges}}} |p|$

prop $\{\text{length} \le k \text{ path } s \rightsquigarrow v\}$

$= \bigcup \{\text{length} \le k-1 \text{ paths } s \rightsquigarrow v\} \bigcup_{(u,v)\in E} \{\text{length} \le k \text{ path } s \to u \to v\}$

$\{p \circ v : p \text{ length} \le k-1 \text{ path } s \rightsquigarrow u\}$

Remark: this is a non-disjoint decompose.

cor $dist_{\le k}(s,t) = \min \begin{cases} dist_{k-1}(s,v) \\ \min_{(u,v)\in E} dist_{\le k-1}(s,u) + C((u,v)) \end{cases}$

prop If no path from s reaching neg cycle, then

$\forall t \quad dist(s,t) = dist_{\le n-1}(s,t)$

prop (Bellman-Ford)

If $\nexists$ no path from s reaches a neg cycle, then $dist(s,t)$ can be computed in $O(nm)$ time.

pf

Algo: (1) for $v \in V$, $d[0][v] = \infty$

(2) $d[0][s] = 0$

(3) for $1 \le k \le n-1$

- for $v \in V$

$$d[k][v] = \min \begin{cases} d[k-1][v] \\ \min_u \left(d[k-1][u] + c((v,u)) \right) \end{cases}$$

Complexity: clearly, $O(n^3)$

Better if we restrict u to only connected. Using adjacency list, we can achieve $O(nm)$.

prop If no path starting from s to neg cycle, then given $\{dist_{\le k}(s,v)\}_{\forall k,v}$ we can compute the shortest path in $O(n+m)$

Doable since path itself is linear.

When neg cycle exists.

prop If $\forall t$ $dist_{\le n-1}(s,t) = dist_{\le n}(s,t)$, then no path from s to neg cycle.

pf: Claim: for all v, $k-1 \ge n$, $dist_{\le k}(s,v) = dist_{\le n-1}(s,t)$

pf by induction on k.

- $k-1=n$, by hypothesis, true.

- $k-1>n$,

$$dist_{\le k}(s,v) = \min \begin{cases} dist_{\le k-1}(s,v) \\ \min\limits_{u} \{ dist_{\le k-1}(s,u) + C((u,v)) \} \end{cases}$$

$$= \min \begin{cases} dist_{\le n-1}(s,v) \\ \min\limits_{u} \{ dist_{\le n-1}(s,u) + C((u,v)) \} \end{cases}$$

$$= dist_{\le n}(s,v) = dist_{\le n-1}(s,v) \quad \square$$

Then, all t, all $k \ge n$, $dist_{\le k}(s,t) = dist_{\le n-1}(s,t)$

$$\Rightarrow \lim_{k\to\infty} dist_{\le k}(s,t) = dist_{\le n-1}(s,t) > -\infty$$

$\Rightarrow$ no neg cycle.

cor: s cannot reach neg cycle iff $\forall t$ $dist_{\le n-1}(s,t) = dist_{\le n}(s,t)$

cor: In $O(nm)$ time, we can compute if
- s reach neg cycle (detect)
- $dist(s,t)$.

Q detect any negative cycle ?

Naive : detect once for each vertex $\Rightarrow O(n \cdot nm)$

Better in $O(nm)$

reduce to 1 instance of Bellmand Ford

Pf: construct G' w/ $V' = V \circ \{s'\}$

$E' = E \circ \{(s', v)\}_{v \in V}$